

# Ruin and risk measures in a bivariate discrete-time risk model

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  - Risk measures
- 3 Bivariate extension of the classical discrete-time model
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  - Numerical example
- 4 Risk measures for the bivariate extension of the classical discrete-time model
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## Section summary

- 1 Introduction
- 2 Classical discrete-time risk model
- 3 Bivariate extension of the classical discrete-time model
- 4 Risk measures for the bivariate extension of the classical discrete-time model

# Background

- In this talk, we suggest an extension of the classical discrete-time risk model in the context of a portfolio assuming two dependant lines of business.
- We develop an algorithm based on Lindley's recursive relation to compute finite time ruin probabilities.
- We then suggest an algorithm to determine the initial surplus an insurer should inject in both lines.

# Motivation

- The principal motivation of this talk is to model each line of business separately, specifying the structure dependence between the lines, instead of modeling directly the sum of both lines of business.
- We could (and we will!) extend the present model in a multivariate context of  $j$  dependent lines of business.

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- 4 Risk measures for the bivariate extension of the classical discrete-time model

## Processes (1/4)

- In this section, we present the classical discrete-time risk model and the assumptions made for the presentation.
- First, we define the aggregate claim amount process. Let  $\underline{W}$  be that process. We have

$$\underline{W} = \{W_n, n \in \mathbb{N}^*\}$$

where  $W_n$  is the aggregate claim amount for the  $n$ -th period.

- We then define the loading factor and loaded premium processes. Let  $\underline{\eta}$  and  $\underline{\pi}$  be, respectively, these processes. We have

$$\underline{\eta} = \{\eta_n, n \in \mathbb{N}^*\}$$

and

$$\underline{\pi} = \{\pi_n, n \in \mathbb{N}^*\}$$

where  $\eta_n$  and  $\pi_n$  are, respectively, the loading factor and the loaded premium for the  $n$ -th period.

- The three last processes are linked by the relation

$$\pi_n = (1 + \eta_n) E[W_n].$$

## Processes (2/4)

- For the presentation, we make the following assumptions :
  - The random variables (r.v.) of the process  $\underline{W}$  are independent and identically distributed (i.i.d.);
  - The loading factor is the same for each period.
- With the last assumptions, we can shorten the notation and use the convention that

$$W_n \sim W \qquad \eta_n = \eta \qquad \pi_n = \pi.$$

- Next, we define the net loss and the cumulative net loss processes. Let  $\underline{L}$  and  $\underline{V}$  be, respectively, these processes. We have

$$\underline{L} = \{L_n, n \in \mathbb{N}\}$$

and

$$\underline{V} = \{V_n, n \in \mathbb{N}\}$$

where  $L_n$  and  $V_n$  are, respectively, the net loss and the cumulative net loss for the  $n$ -th period, with  $L_0 = 0$  and  $V_0 = 0$ .



## Processes (3/4)

- We can express the last two processes in terms of  $\underline{W}$  and  $\eta$  by the relations

$$L_n = W_n - \pi$$

and

$$V_n = \sum_{i=1}^n L_i.$$

- The last process we define is the maximum reach by the cumulative net loss. Let  $\underline{Z}$  be this process. We have

$$\underline{Z} = \{Z_n, n \in \mathbb{N}\}$$

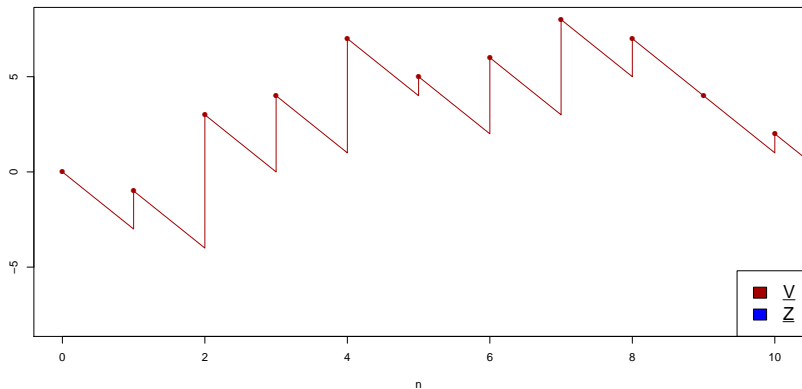
where  $Z_n$  is the maximum reach by the cumulative net loss at the end of the  $n$ -th period.

- We can write  $\underline{Z}$  in terms of  $\underline{L}$  with the relation

$$Z_n = \max_{i \in \{0,1,\dots,n\}} V_i = \max_{i \in \{0,1,\dots,n\}} \left( \sum_{j=0}^i L_j \right). \quad (1)$$

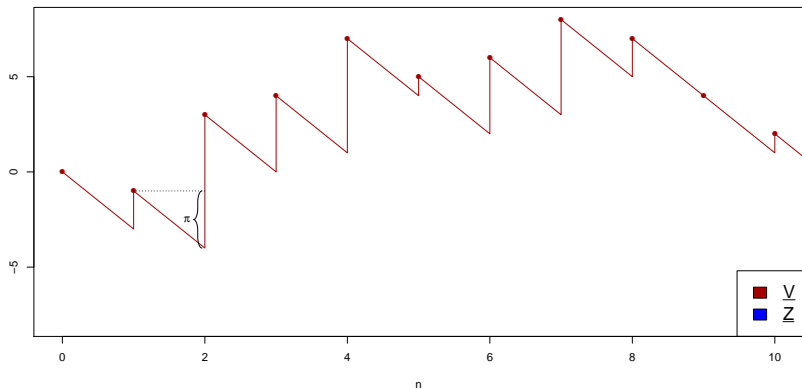
# Processes (4/4)

- We illustrate the processes we defined in the current section.



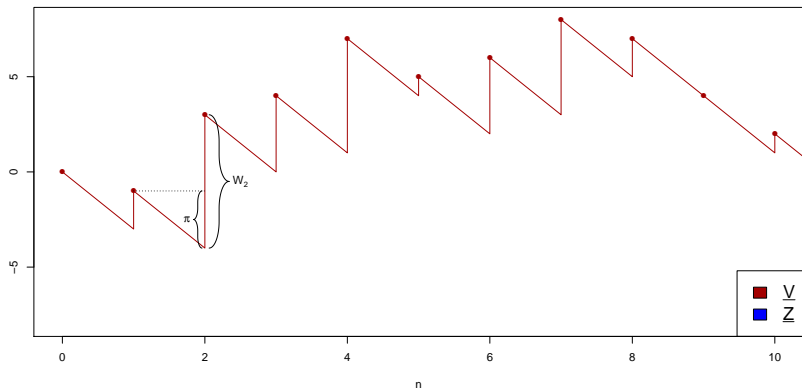
## Processes (4/4)

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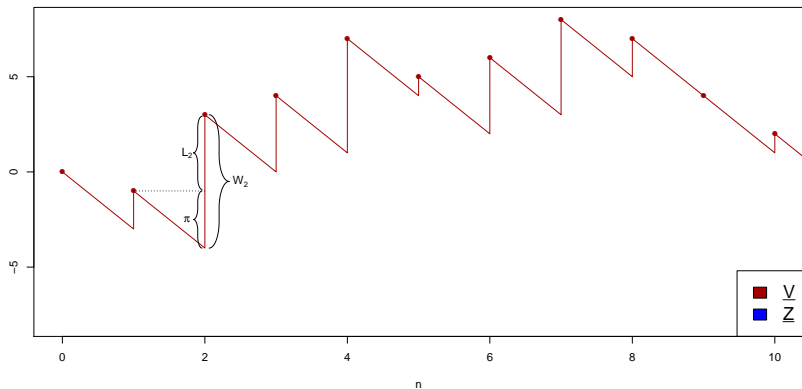
## Processes (4/4)

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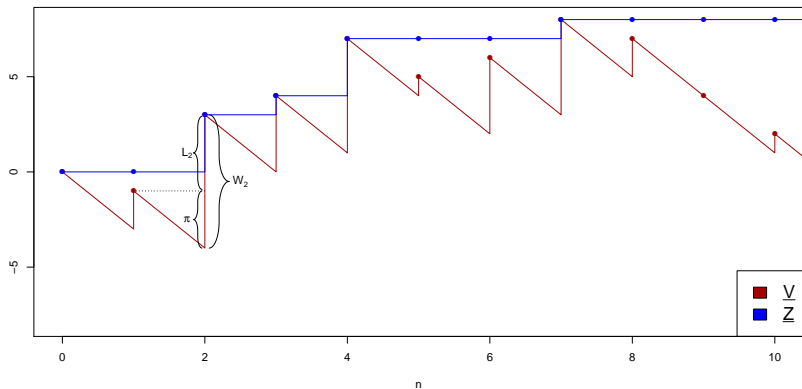
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## Processes (4/4)

- We illustrate the processes we defined in the current section.



## Time of ruin (1/2)

- Our ultimate goal is to calculate a ruin probability for the model. Following this idea, we first define the time of ruin. Let  $\tau$  be this time of ruin. Using the processes we previously define, we have

$$\tau(u) = \begin{cases} \operatorname{argmin}_{n \in \mathbb{N}} (Z_n > u) & , \quad \text{if } \max_{n \in \mathbb{N}} (Z_n) > u \\ \infty & , \quad \text{else} \end{cases}$$

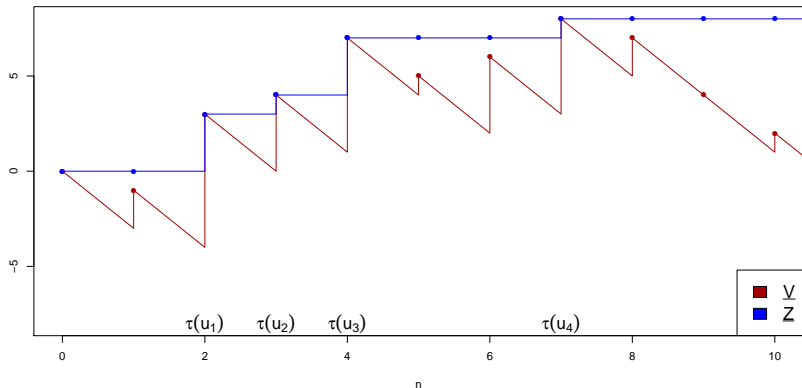
where  $u$  ( $u \geq 0$ ) is the initial capital injected by the insurer.

- In this presentation, we are interested in the finite-time ruin probability over  $n$  periods. We define it by

$$\begin{aligned} \Psi(u, n) &= \Pr[\tau(u) \leq n] \\ &= \Pr[Z_n > u] \end{aligned}$$

## Time of ruin (2/2)

- We illustrate the time of ruin on the following picture.



where  $0 \leq u_1 < 3 \leq u_2 < 4 \leq u_3 < 7 \leq u_4 < 8$ .



## Traditionnal method

- In this subsection, we will look at two different ways to evaluate the finite-time ruin probability defined in the last subsection.
- First of all, we rewrite the ruin probability as

$$\begin{aligned}\Psi(u, n) &= \Pr[Z_n > u] \\ &= 1 - F_{Z_n}(u).\end{aligned}$$

- As we can see, if we know the distribution of  $Z_n$ , we know the ruin probability over that  $n$ -period time for a given initial capital.
- We can easily compute the c.d.f. of  $Z_n$  by using a recursive formula. We have

$$F_{Z_n}(x) = \sum_{i=0}^{\lfloor x \rfloor + \pi} \Pr[W = i] F_{Z_{n-1}}(x + \pi - i)$$

with  $F_{Z_0}(x) = 1$  for  $x \in \mathbb{R}_+$ .

## Lindley's recursive relation (1/5)

- We now look at an alternative way to calculate the ruin probability based on Lindley's recursive relation.
- First, we define a new process  $\tilde{Z} = \{\tilde{Z}_k, k \in \mathbb{N}\}$  where

$$\begin{aligned}
 \tilde{Z}_n &= V_n - \min_{i \in \{0,1,\dots,n\}} (V_i) \\
 &= \max_{i \in \{0,1,\dots,n\}} (V_n - V_i) \\
 &= \max_{i \in \{0,1,\dots,n\}} \left( \sum_{j=0}^n L_j - \sum_{j=0}^i L_j \right) \\
 &= \max_{i \in \{0,1,\dots,n\}} \left( \sum_{j=i+1}^n L_j \right). \tag{2}
 \end{aligned}$$

## Lindley's recursive relation (2/5)

### Proposition.

Since the random variables in  $\underline{L}$  are i.i.d., each random variables in  $\underline{Z}$  and  $\tilde{\underline{Z}}$  are equal in distribution.

### Proof.

First, expanding  $Z_n$  by recalling (1), we obtain

$$Z_n = \max \left\{ \underbrace{0}_{0 \text{ r.v.}}, \underbrace{L_1}_{1 \text{ r.v.}}, \underbrace{L_1 + L_2}_{2 \text{ r.v.}}, \dots, \underbrace{L_1 + L_2 + \dots + L_n}_{n \text{ r.v.}} \right\}.$$

Second, expanding  $\tilde{Z}_n$  by recalling (2), we obtain

$$\tilde{Z}_n = \max \left\{ \underbrace{L_1 + L_2 + \dots + L_n}_{n \text{ r.v.}}, \underbrace{L_2 + L_3 + \dots + L_n}_{(n-1) \text{ r.v.}}, \dots, \underbrace{L_n}_{1 \text{ r.v.}}, \underbrace{0}_{0 \text{ r.v.}} \right\}.$$

Then, since the  $L_n$  are i.i.d.,  $Z_n$  and  $\tilde{Z}_n$  have the same distribution. □

## Lindley's recursive relation (3/5)

Proposition.

In the present model, the dynamic of the process  $\tilde{Z}$  is given by

$$\tilde{Z}_n = \max \left( \tilde{Z}_{n-1} + L_n, 0 \right), n \in \mathbb{N}^*.$$

Proof.

$$\begin{aligned} \max \left( \tilde{Z}_{n-1} + L_n, 0 \right) &= \max \left( \max_{i \in \{0,1,\dots,n-1\}} \left( \sum_{j=i+1}^{n-1} L_j \right) + L_n, 0 \right) \\ &= \max \left( \max_{i \in \{0,1,\dots,n-1\}} \left( \sum_{j=i+1}^n L_j \right), 0 \right) \\ &= \max_{i \in \{0,1,\dots,n\}} \left( \sum_{j=i+1}^n L_j \right) \\ &= \tilde{Z}_n \end{aligned}$$



## Lindley's recursive relation (4/5)

- We use the two last propositions to compute the finite-time ruin probability.
- First, we have

$$\begin{aligned}
 \Pr \left[ \tilde{Z}_n = k \right] &= \Pr \left[ \max \left( \tilde{Z}_{n-1} + L_n, 0 \right) = k \right] \\
 &= \sum_{i=0}^{k+\pi} \Pr \left[ \max \left( \tilde{Z}_{n-1} + L_n, 0 \right) = k \mid \tilde{Z}_{n-1} = i \right] \Pr \left[ \tilde{Z}_{n-1} = i \right] \\
 &= \sum_{i=0}^{k+\pi} \zeta(k, i) \Pr \left[ \tilde{Z}_{n-1} = i \right]
 \end{aligned}$$

where

$$\begin{aligned}
 \zeta(k, i) &= \begin{cases} \Pr[i + L_n \leq 0] & , \quad k = 0 \\ \Pr[i + L_n = k] & , \quad k \in \mathbb{N}^* \end{cases} \\
 &= \begin{cases} F_W(\pi - i) & , \quad k = 0 \\ \Pr[W = k + \pi - i] & , \quad k \in \mathbb{N}^* \end{cases}
 \end{aligned}$$

and  $\Pr \left[ \tilde{Z}_0 = 0 \right] = 1$ .

## Lindley's recursive relation (5/5)

- We then easily compute the c.d.f. of  $Z_n$  with the relation

$$F_{Z_n}(x) = \sum_{i=0}^{\lfloor x \rfloor} \Pr \left[ \tilde{Z}_n = i \right].$$

- We then have

$$\Psi(u, n) = 1 - F_{Z_n}(u).$$

## Dynamic Value-at-Risk

- We now introduce the dynamic Value-at-Risk.
- We denote it by  $\text{VaR}_\kappa(Z_n)$ , where

$$\text{VaR}_\kappa(Z_n) = F_{Z_n}^{-1}(\kappa)$$

and

$$F_{Z_n}^{-1}(\kappa) = \inf_{x \in \mathbb{R}} \{F_{Z_n}(x) \geq \kappa\}.$$

- In other words,  $\text{VaR}_\kappa(Z_n)$  is the minimal initial surplus for which the ruin probability over  $n$  periods is lower than  $(1 - \kappa)$ .

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## Processes

- In this section, we introduce an extension of the model in which there is two dependant lines of business.
- We use exactly the same notation as in the previous section, except that we add an upper index representing the business line for each process.
- For example, the bivariate aggregate claims process is represented by

$$(\underline{W}^{(1)}, \underline{W}^{(2)}).$$

- We still make the assumptions that each couple in  $(\underline{W}^{(1)}, \underline{W}^{(2)})$  are i.i.d. and that the loading factor (which can be different for each business line) are constant for every period.
- Note :  $W_1^{(1)}$  and  $W_1^{(2)}$  are dependent but  $W_1^{(1)}$  and  $W_2^{(1)}$  are independent.

## Time of ruin (1/2)

- Even if the processes are defined the same in the univariate and the bivariate model, the definition of ruin is different among the two models.
- With two lines of business, we can extend the concept of ruin probability. Indeed, we can be interested by the ruin of a single line of business or by the ruin of the two lines of business.
- Let  $\Psi_{\text{and}}(u^{(1)}, u^{(2)}, n)$  and  $\Psi_{\text{or}}(u^{(1)}, u^{(2)}, n)$  be, respectively, the ruin probability for the two lines of business and the ruin probability for at least one of the lines of business over  $n$  periods and for which we have initial surplus of  $(u^{(1)}, u^{(2)})$ .
- Then, we have

$$\Psi_{\text{and}}(u^{(1)}, u^{(2)}, n) = \Pr[\tau_{\text{and}}(u^{(1)}, u^{(2)}) \leq n]$$

$$\Psi_{\text{or}}(u^{(1)}, u^{(2)}, n) = \Pr[\tau_{\text{or}}(u^{(1)}, u^{(2)}) \leq n]$$

where

$$\tau_{\text{and}}(u^{(1)}, u^{(2)}) = \max(\tau^{(1)}(u^{(1)}), \tau^{(2)}(u^{(2)}))$$

$$\tau_{\text{or}}(u^{(1)}, u^{(2)}) = \min(\tau^{(1)}(u^{(1)}), \tau^{(2)}(u^{(2)})).$$

## Time of ruin (2/2)

- As it was the case for the univariate model, the strategy to rewrite these two ruin probabilities in terms of the multivariate c.d.f. of  $(Z^{(1)}, Z^{(2)})$ .
- We then have

$$\begin{aligned}\Psi_{\text{and}}(u^{(1)}, u^{(2)}, n) &= \Pr [Z_n^{(1)} > u^{(1)} \cap Z_n^{(2)} > u^{(2)}] \\ &= 1 - F_{Z_n^{(1)}}(u^{(1)}) - F_{Z_n^{(2)}}(u^{(2)}) + F_{Z_n^{(1)}, Z_n^{(2)}}(u^{(1)}, u^{(2)})\end{aligned}\quad (3)$$

and

$$\begin{aligned}\Psi_{\text{or}}(u^{(1)}, u^{(2)}, n) &= \Pr [Z_n^{(1)} > u^{(1)} \cup Z_n^{(2)} > u^{(2)}] \\ &= 1 - F_{Z_n^{(1)}, Z_n^{(2)}}(u^{(1)}, u^{(2)})\end{aligned}\quad (4)$$

## Lindley's recursive relation (1/3)

- For the bivariate model, we will use an extension of the method based on Lindley's recursive relation presented in the previous section to compute the c.d.f. of  $Z_n$ .
- We then introduce the process  $(\tilde{Z}_n^{(1)}, \tilde{Z}_n^{(2)})$ , which is defined the same as in the previous section.
- Since the couples  $(L_n^{(1)}, L_n^{(2)})$  are i.i.d.,  $(Z_n^{(1)}, Z_n^{(2)})$  and  $(\tilde{Z}_n^{(1)}, \tilde{Z}_n^{(2)})$  are equal in distribution.
- The idea of the proof is the same as it was in the previous section.

## Lindley's recursive relation (2/3)

- Then, we have

$$\begin{aligned} \Pr \left[ \tilde{Z}_n^{(1)} = k_1, \tilde{Z}_n^{(2)} = k_2 \right] &= \Pr \left[ \max \left( \tilde{Z}_{n-1}^{(1)} + L_n^{(1)}, 0 \right) = k_1, \max \left( \tilde{Z}_{n-1}^{(2)} + L_n^{(2)}, 0 \right) = k_2 \right] \\ &= \sum_{i_1=0}^{k_1 + \pi^{(1)}} \sum_{i_2=0}^{k_2 + \pi^{(2)}} \zeta(k_1, k_2, i_1, i_2) \Pr \left[ \tilde{Z}_{n-1}^{(1)} = i_1, \tilde{Z}_{n-1}^{(2)} = i_2 \right] \end{aligned}$$

where

$$\zeta(k_1, k_2, i_1, i_2) = \begin{cases} F_{W^{(1)}, W^{(2)}}(\pi^{(1)} - i_1, \pi^{(2)} - i_2) & , (k_1, k_2) = (0, 0) \\ F_{W^{(1)}=k_1 + \pi^{(1)} - i_1, W^{(2)}}(\pi^{(2)} - i_2) & , (k_1, k_2) \in (\mathbb{N}^*, 0) \\ F_{W^{(1)}, W^{(2)}=k_2 + \pi^{(2)} - i_2}(\pi^{(1)} - i_1) & , (k_1, k_2) \in (0, \mathbb{N}^*) \\ \Pr \left[ W^{(1)} = k_1 + \pi^{(1)} - i_1, W^{(2)} = k_2 + \pi^{(2)} - i_2 \right] & , (k_1, k_2) \in \mathbb{N}^{*2} \end{cases}$$

$$\text{and } \Pr \left[ \tilde{Z}_0^{(1)} = 0, \tilde{Z}_0^{(2)} = 0 \right] = 1.$$

## Lindley's recursive relation (3/3)

- We then easily compute the c.d.f. of  $(Z_n^{(1)}, Z_n^{(2)})$  with the relation

$$F_{Z_n^{(1)}, Z_n^{(2)}}(x_1, x_2) = \sum_{i_1=0}^{\lfloor x_1 \rfloor} \sum_{i_2=0}^{\lfloor x_2 \rfloor} \Pr \left[ \tilde{Z}_n^{(1)} = i_1, \tilde{Z}_n^{(2)} = i_2 \right].$$

- We then use relation (3) and (4) to compute the desired ruin probabilities.

## Hypothesis

- Let  $W^{(1)} \sim \text{Poisson}(\lambda = 1.8)$  and  $W^{(2)} \sim \text{NegBin}(r = 4, q = 0.6)$ .
- The dependence between  $W^{(1)}$  and  $W^{(2)}$  is modeled by a Frank copula with parameter  $\alpha$  where

$$C_{\alpha}(u_1, u_2) = -\frac{1}{\alpha} \ln \left( 1 + \frac{(e^{-\alpha u_1} - 1)(e^{-\alpha u_2} - 1)}{e^{-\alpha} - 1} \right), \quad \alpha \neq 0.$$

- The loading factors are  $\eta^{(1)} = \frac{1}{9}$  and  $\eta^{(2)} = \frac{1}{8}$ .
- The loaded premiums are  $\pi^{(1)} = 2$  and  $\pi^{(2)} = 3$ .
- We are interested in the two ruin probabilities over 12 periods.
- We can see the effect of the dependence parameter  $\alpha$  on the following illustrations.

## Results (1/2)

- We illustrate the two ruin probabilities for different levels of initial surplus.
- $\alpha = -5$

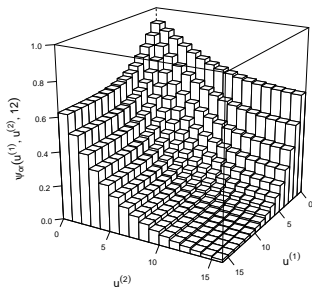
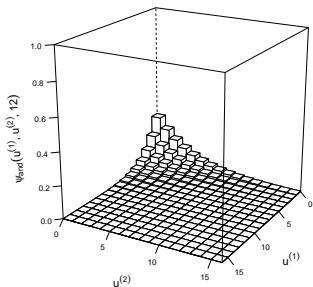


Figure: Illustration of  $\Psi_{\text{and}}(u^{(1)}, u^{(2)}, 12)$  (left) and  $\Psi_{\text{or}}(u^{(1)}, u^{(2)}, 12)$  (right).



## Results (1/2)

- We illustrate the two ruin probabilities for different levels of initial surplus.
- $\alpha = -1$

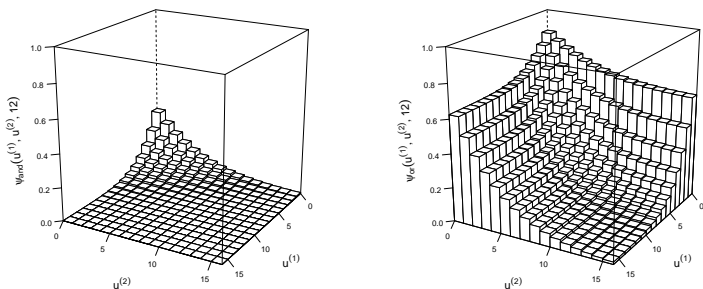


Figure: Illustration of  $\Psi_{\text{and}}(u^{(1)}, u^{(2)}, 12)$  (left) and  $\Psi_{\text{or}}(u^{(1)}, u^{(2)}, 12)$  (right).

## Results (1/2)

- We illustrate the two ruin probabilities for different levels of initial surplus.
- $\alpha = 1$

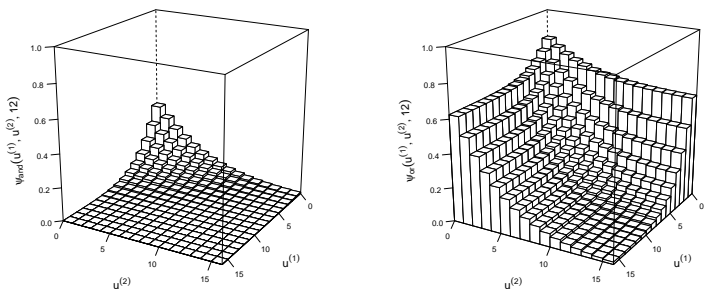


Figure: Illustration of  $\Psi_{\text{and}}(u^{(1)}, u^{(2)}, 12)$  (left) and  $\Psi_{\text{or}}(u^{(1)}, u^{(2)}, 12)$  (right).

## Results (1/2)

- We illustrate the two ruin probabilities for different levels of initial surplus.
- $\alpha = 5$

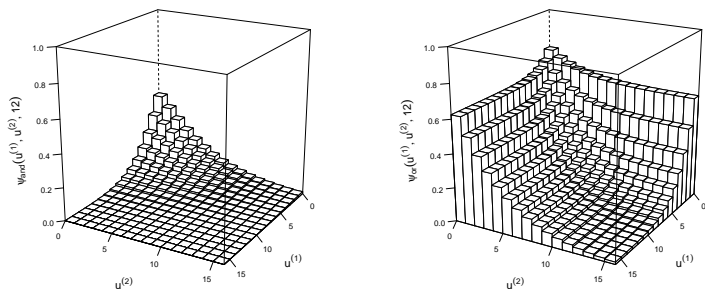


Figure: Illustration of  $\Psi_{\text{and}}(u^{(1)}, u^{(2)}, 12)$  (left) and  $\Psi_{\text{or}}(u^{(1)}, u^{(2)}, 12)$  (right).

## Results (2/2)

- When  $\alpha$  increases,  $\Psi_{\text{and}}(u^{(1)}, u^{(2)}, n)$  increases.
- When  $\alpha$  increases,  $\Psi_{\text{or}}(u^{(1)}, u^{(2)}, n)$  decreases.
- These results are instinctive.
- When dependence between lines of business is greater, if one line of business has a problem or is going well, then the other one is more likely to follow.

### Remarks

- One could argue that  $\lim_{u^{(i)} \rightarrow \infty} \Psi_{\text{and}}(u^{(1)}, u^{(2)}, n) = 0, i \in \{1, 2\}$ . Clearly, when the initial surplus of one line of business is going to infinity, this line of business can't ruin, making impossible that both lines of business ruin.
- One could argue that  $\lim_{u^{(i)} \rightarrow \infty} \Psi_{\text{or}}(u^{(1)}, u^{(2)}, n) = \Psi(u^{(j)}, n), (i, j) \in \{1, 2\}^2, i \neq j$ . Again, this line of business can't ruin, making the probability that at least one line of business ruins equal directly to the marginal ruin probability of the other line of business.

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## Optimal initial surplus (1/8)

- Here, we study an extended risk measure of the univariate dynamic VaR.
- In the univariate case, the dynamic VaR was giving us the minimal initial surplus needed to keep the ruin probability below  $(1 - \kappa)$ .
- We use the same approach here, but with both the ruin probability "and" and "or".
- First, we can define two different sets containing every couple which meet this condition individually. We then have

$$\mathcal{D}_{n;\kappa}^{(\text{and})} = \left\{ (u^{(1)}, u^{(2)}) : \Psi_{\text{and}}(u^{(1)}, u^{(2)}, n) \leq 1 - \kappa \right\}$$

and

$$\mathcal{D}_{n;\kappa}^{(\text{or})} = \left\{ (u^{(1)}, u^{(2)}) : \Psi_{\text{or}}(u^{(1)}, u^{(2)}, n) \leq 1 - \kappa \right\}.$$

- One can easily show that  $\mathcal{D}_{n;\kappa}^{(\text{or})} \subseteq \mathcal{D}_{n;\kappa}^{(\text{and})}$  using the fact that  $\Psi_{\text{or}}(u^{(1)}, u^{(2)}, n) \leq \Psi_{\text{and}}(u^{(1)}, u^{(2)}, n)$ .

## Optimal initial surplus (2/8)

- We illustrate  $\mathcal{D}_{n;\kappa}^{(\text{and})}$  and  $\mathcal{D}_{n;\kappa}^{(\text{or})}$ .

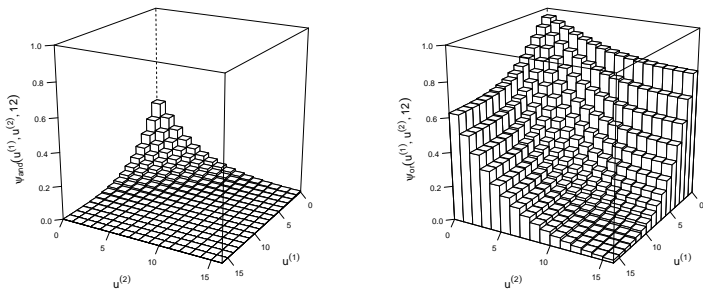


Figure: Illustration of  $\mathcal{D}_{12;0.85}^{(\text{and})}$  (left) and  $\mathcal{D}_{12;0.85}^{(\text{or})}$  (right).

## Optimal initial surplus (2/8)

- We illustrate  $\mathcal{D}_{n;\kappa}^{(\text{and})}$  and  $\mathcal{D}_{n;\kappa}^{(\text{or})}$ .

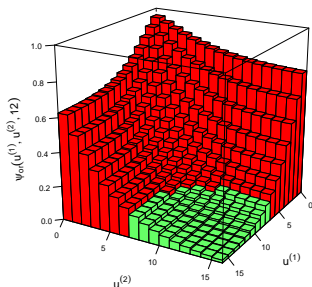
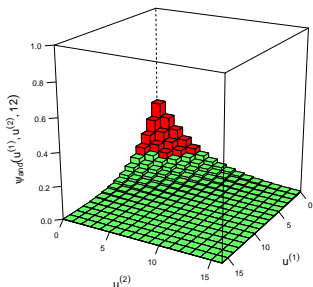


Figure: Illustration of  $\mathcal{D}_{12;0.85}^{(\text{and})}$  (left) and  $\mathcal{D}_{12;0.85}^{(\text{or})}$  (right).



## Optimal initial surplus (3/8)

- Although the ruin probability of each couple in  $\mathcal{D}_{n;\kappa}^{(\text{and})}$  and  $\mathcal{D}_{n;\kappa}^{(\text{or})}$  is lower than  $(1 - \kappa)$ , the majority of them are not efficient.
- The next step is then to limit these sets to their respective efficient subset  $\mathcal{D}'_{n;\kappa}^{(\text{and})}$  and  $\mathcal{D}'_{n;\kappa}^{(\text{or})}$ .
- Then, we have

$$\mathcal{D}'_{n;\kappa}^{(\text{and})} = \left\{ \left\{ (u^{(1)} - 1, u^{(2)}), (u^{(1)}, u^{(2)} - 1) \right\} \cap \mathcal{D}_{n;\kappa}^{(\text{and})} = \emptyset : (u^{(1)}, u^{(2)}) \in \mathcal{D}_{n;\kappa}^{(\text{and})} \right\}$$

and

$$\mathcal{D}'_{n;\kappa}^{(\text{or})} = \left\{ \left\{ (u^{(1)} - 1, u^{(2)}), (u^{(1)}, u^{(2)} - 1) \right\} \cap \mathcal{D}_{n;\kappa}^{(\text{or})} = \emptyset : (u^{(1)}, u^{(2)}) \in \mathcal{D}_{n;\kappa}^{(\text{or})} \right\}.$$

## Optimal initial surplus (4/8)

- We illustrate  $\mathcal{D}'_{n;\kappa}^{(\text{and})}$  and  $\mathcal{D}'_{n;\kappa}^{(\text{or})}$ .

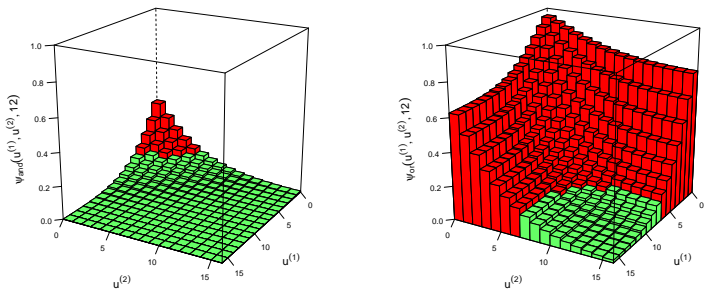


Figure: Illustration of  $\mathcal{D}'_{12;0.85}^{(\text{and})}$  (left) and  $\mathcal{D}'_{12;0.85}^{(\text{or})}$  (right).

## Optimal initial surplus (4/8)

- We illustrate  $\mathcal{D}'_{n;\kappa}^{(\text{and})}$  and  $\mathcal{D}'_{n;\kappa}^{(\text{or})}$ .

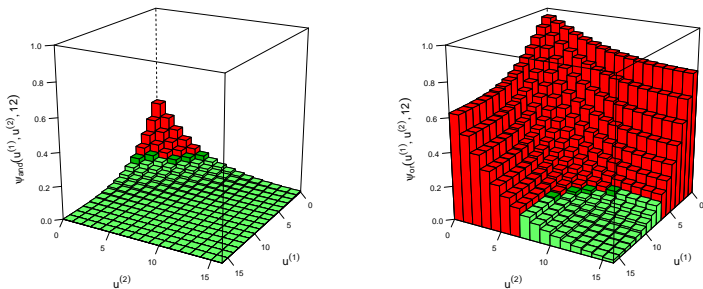


Figure: Illustration of  $\mathcal{D}'_{12;0.85}^{(\text{and})}$  (left) and  $\mathcal{D}'_{12;0.85}^{(\text{or})}$  (right).

## Optimal initial surplus (5/8)

- Now that we have the efficient subsets, we will restrain them again by choosing the couples in the efficient subsets for which the insurer will have the lowest initial cost.
- These couples are represented by the lowest diagonal containing at least one couple.
- Then, we have

$$\mathcal{D}_{n;\kappa}''(\text{and}) = \underset{(u^{(1)}, u^{(2)}) \in \mathcal{D}_{n;\kappa}'(\text{and})}{\operatorname{argmin}} (u^{(1)} + u^{(2)})$$

and

$$\mathcal{D}_{n;\kappa}''(\text{or}) = \underset{(u^{(1)}, u^{(2)}) \in \mathcal{D}_{n;\kappa}'(\text{or})}{\operatorname{argmin}} (u^{(1)} + u^{(2)}).$$

## Optimal initial surplus (6/8)

- We illustrate  $\mathcal{D}_{n;\kappa}^{\prime\prime(\text{and})}$  and  $\mathcal{D}_{n;\kappa}^{\prime\prime(\text{or})}$ .

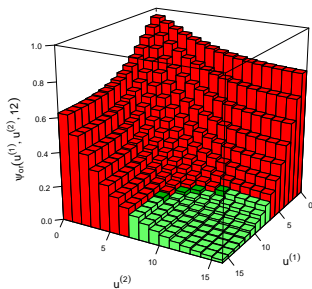
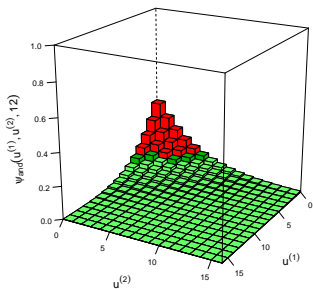


Figure: Illustration of  $\mathcal{D}_{12;0.85}^{\prime\prime(\text{and})}$  (left) and  $\mathcal{D}_{12;0.85}^{\prime\prime(\text{or})}$  (right).

## Optimal initial surplus (6/8)

- We illustrate  $\mathcal{D}_{n;\kappa}^{//(and)}$  and  $\mathcal{D}_{n;\kappa}^{//(or)}$ .

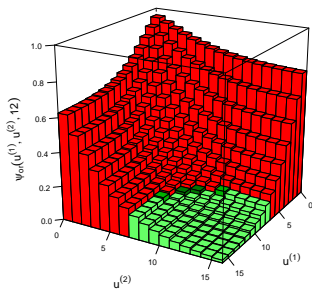
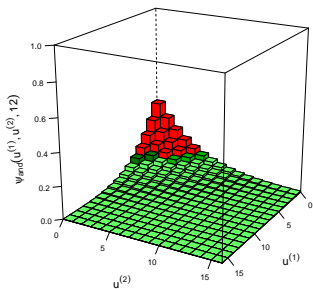


Figure: Illustration of  $\mathcal{D}_{12;0.85}^{//(and)}$  (left) and  $\mathcal{D}_{12;0.85}^{//(or)}$  (right).

## Optimal initial surplus (7/8)

- The couples in the subsets  $\mathcal{D}_{n;\kappa}^{''(\text{and})}$  and  $\mathcal{D}_{n;\kappa}^{''(\text{or})}$  need the same initial surplus for the insurer.
- The different couples are just different ways to allocate the initial surplus among the two lines of business. The insurer will logically choose the one which is minimizing his selected ruin probability.
- Then, we have

$$\mathcal{D}_{n;\kappa}^{*(\text{and})} = \underset{(u^{(1)}, u^{(2)}) \in \mathcal{D}_{n;\kappa}^{''(\text{and})}}{\operatorname{argmin}} \left( \Psi_{\text{and}} \left( u^{(1)}, u^{(2)}, n \right) \right)$$

and

$$\mathcal{D}_{n;\kappa}^{*(\text{or})} = \underset{(u^{(1)}, u^{(2)}) \in \mathcal{D}_{n;\kappa}^{''(\text{or})}}{\operatorname{argmin}} \left( \Psi_{\text{or}} \left( u^{(1)}, u^{(2)}, n \right) \right).$$

## Optimal initial surplus (8/8)

- We illustrate  $\mathcal{D}_{n;\kappa}^{*(\text{and})}$  and  $\mathcal{D}_{n;\kappa}^{*(\text{or})}$ .

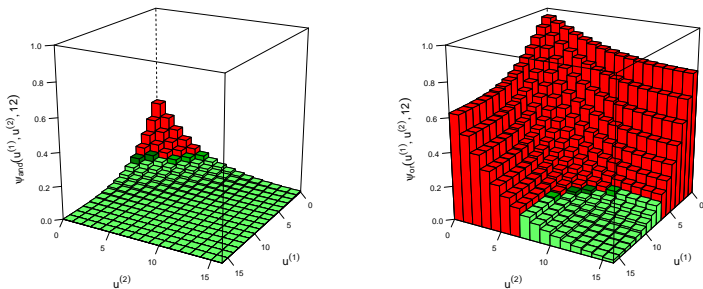


Figure: Illustration of  $\mathcal{D}_{12;0.85}^{*(\text{and})}$  (left) and  $\mathcal{D}_{12;0.85}^{*(\text{or})}$  (right).



## Optimal initial surplus (8/8)

- We illustrate  $\mathcal{D}_{n;\kappa}^{*(\text{and})}$  and  $\mathcal{D}_{n;\kappa}^{*(\text{or})}$ .

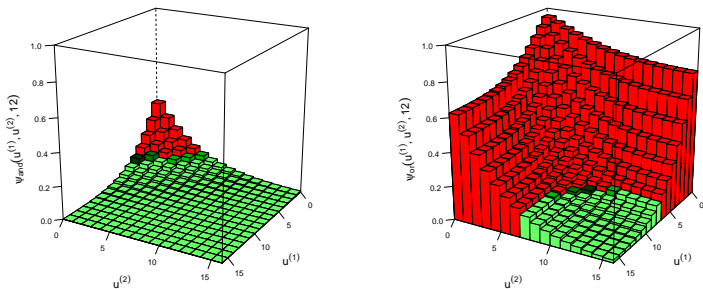


Figure: Illustration of  $\mathcal{D}_{12;0.85}^{*(\text{and})}$  (left) and  $\mathcal{D}_{12;0.85}^{*(\text{or})}$  (right).

## Improvements

- With the present approach, the insurer can choose to keep one of the ruin probability under  $(1 - \kappa)$ .
- It could be interesting to combine the two ruin probabilities and find the optimal way to allocate the initial surplus while keeping the ruin probability "and" below  $(1 - \kappa_{\text{and}})$  and the ruin probability "or" below  $(1 - \kappa_{\text{or}})$ .

# Hypothesis

- We use the same numerical example than in the previous section.
- We want to see the impact of the dependence parameter  $\alpha$  on the initial surplus that will choose to inject the insurer.

## Results (1/3)

- We illustrate the chosen initial surplus for each ruin probability.
- $\alpha = -5$

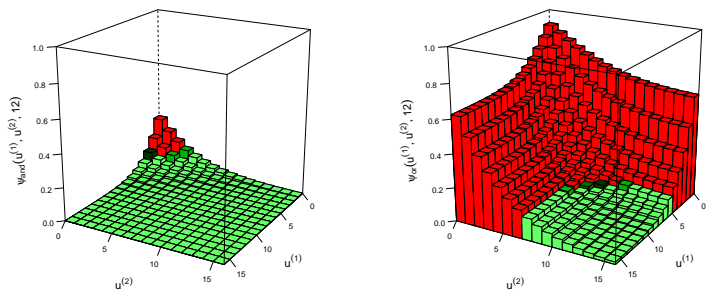


Figure: Illustration of  $\Psi_{\text{and}}(u^{(1)}, u^{(2)}, 12)$  (left) and  $\Psi_{\text{or}}(u^{(1)}, u^{(2)}, 12)$  (right).

## Results (1/3)

- We illustrate the chosen initial surplus for each ruin probability.
- $\alpha = -1$

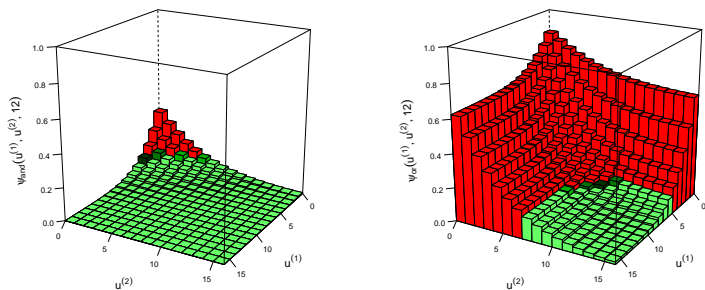


Figure: Illustration of  $\Psi_{\text{and}}(u^{(1)}, u^{(2)}, 12)$  (left) and  $\Psi_{\text{or}}(u^{(1)}, u^{(2)}, 12)$  (right).

## Results (1/3)

- We illustrate the chosen initial surplus for each ruin probability.
- $\alpha = 1$

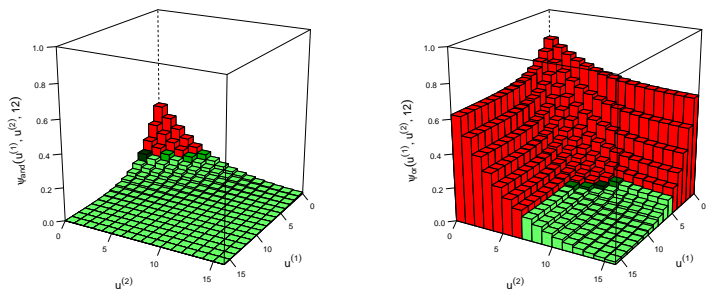


Figure: Illustration of  $\Psi_{\text{and}}(u^{(1)}, u^{(2)}, 12)$  (left) and  $\Psi_{\text{or}}(u^{(1)}, u^{(2)}, 12)$  (right).

## Results (1/3)

- We illustrate the chosen initial surplus for each ruin probability.
- $\alpha = 5$

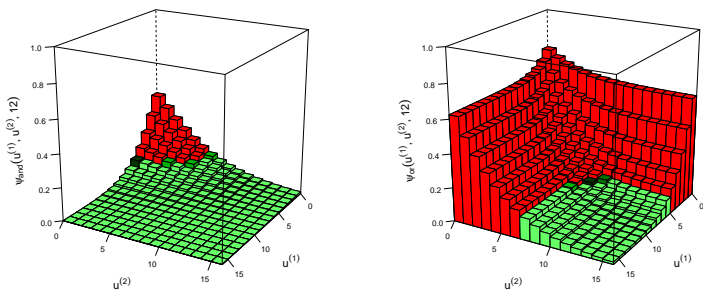


Figure: Illustration of  $\Psi_{\text{and}}(u^{(1)}, u^{(2)}, 12)$  (left) and  $\Psi_{\text{or}}(u^{(1)}, u^{(2)}, 12)$  (right).

## Results (2/3)

- The results depending on  $\kappa$  and  $\alpha$  are presented in the following table :

Table: Some values of  $\mathcal{D}_{12;\kappa}^{*(\text{and})}$ .

$\kappa$	$\alpha$			
	-5	-1	1	5
0.85	( 2, 0)	( 3, 0)	( 3, 0)	( 4, 0)
0.90	( 3, 0)	( 4, 0)	( 5, 0)	( 5, 0)
0.95	( 5, 0)	( 6, 0)	( 6, 0)	( 7, 0)
0.99	( 8, 0)	( 9, 0)	( 9, 0)	(10, 0)



## Results (2/3)

- The results depending on  $\kappa$  and  $\alpha$  are presented in the following table :

Table: Some values of  $\mathcal{D}_{12;\kappa}^{*(or)}$ .







$\kappa$	$\alpha$			
	-5	-1	1	5
0.85	( 6, 9)	( 6, 9)	( 6, 9)	( 6, 8)
0.90	( 7,10)	( 7,10)	( 7,10)	( 7,10)
0.95	( 9,12)	( 9,12)	( 9,12)	( 9,12)
0.99	(12,17)	(12,17)	(11,18)	(11,18)

## Results (3/3)

- The best way to allocate the initial surplus to keep the ruin probability "and" below  $(1 - \kappa)$  is to give all the capital to the less risky line of business.
- We only need to minimize the ruin probability of one line of business to minimize the "and" probability.
- When we work with the "or" probability, we need to manage the initial surplus among the two lines of business.
- Clearly, in this case, minimizing the ruin probability of only one line of business won't minimize the "or" probability.

Thank you for your attention!

## References

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